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TORSIONAL VIBRATIONS OF THE DRIVE SHAFTS OF MECHANISMS

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A study on the torsional vibrations of the drive shafts of mechanisms having variable inertia has been presented in a previous paper by the authors. The drive shaft has been assumed to be a continuous element with distributed mass. Continuous element modelling of the shaft leads to a different mathematical formulation. In this previous paper, the boundary value problem of torsional vibrations of a drive shaft has been formulated and the forced response has been examined. In the analysis, a simpler form of boundary condition for the motor side of the shaft has been considered. In this paper, the analysis is extended to a more general boundary condition of the motor side of the shaft. For this purpose, two solution techniques, a series solution method and a perturbation solution, are given and the effects of motor characteristics on the behavior of the drive shaft are examined.

1. INTRODUCTION

Mechanisms cause parametrically excited torsional vibrations in their shafts due to variable inertia. This problem, being of the greatest importance in the design of motors and machines, has been investigated by many researchers. In these studies, crankshafts, and input and output shafts, of various systems are modelled as discrete systems and the governing equations of motions are described by ordinary differential equations with periodic coefficients. A detailed stability analysis of the systems has explained fractures in the shafts, which have been attributed to the phenomenon of torsional vibration with variable inertia [1–8].

In a previous paper [9], the authors have investigated the torsional vibrations of the drive shafts of mechanisms having variable inertia characteristics by modelling the shaft as a continuous element. The governing equation of the torsional vibration of the shaft has been obtained as a partial differential equation and the forced response has been examined by linearized formulation. A perturbation method has been adapted to the problem and the resonance conditions have been exposed. Two different boundary conditions, a simpler one and a general one, have been given for the motor side of the shaft. The analysis has been based on the simpler case. It has been pointed out that, for the general case this analysis has to be modified.

In the present paper, the theoretical analysis is extended to a more complex boundary condition on the motor side of the shaft. If the solution procedure is applied in the same manner as for the simplier case, then secular terms appear during the recursive solutions. In order to eliminate these terms, the perturbation solution has to be modified. In that case, it is convenient to choose the average angular velocity used in the linearization

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procedure of the problem as a free parameter around the nominal angular velocity. In addition to this method, a direct solution method called the series solution is given for the forced response. This method is also an extended version of the standard Fourier analysis used for determination of the forced response of an ordinary differential equation with periodic coefficients and under a forcing function. It is mainly used to examine the effects of motor characteristics on the behavior of the drive shaft.

2. EQUATION OF MOTION

The system considered consists of an asynchronous motor a coupling shaft and a mechanism (see Figure 1). The physical system is chosen to be relatively simple in order to avoid theoretical difficulties. Since the shaft is assumed to be a continuous element, the governing equation of the motion becomes a classical wave equation.

The partial differential equation of torsional vibrations of the shaft is

$$\frac{\partial^2 \varphi(x, t)}{\partial t^2} = c^2 \frac{\partial^2 \varphi(x, t)}{\partial x^2}, \tag{1}$$

where $c^2 = G/\rho$.

The boundary conditions of the motor and mechanism sides of the shaft are as follows:

$$J_0 \ddot{\varphi}(0, t) - GI_p \varphi'(0, t) = M_m, \tag{2}$$

$$J[\varphi(l, t)]\ddot{\varphi}(l, t) + \tfrac{1}{2}\frac{\mathrm{d}J[\varphi(l, t)]}{\mathrm{d}\varphi(l, t)}\dot{\varphi}^2(l, t) + GI_p \varphi'(l, t) = -M_w[\varphi(l, t), \dot{\varphi}(l, t), t], \tag{3}$$

Here, M_m and J_0 are the motor torque and the mass moment of inertia; G, l_p, ρ and l are the shear modulus, the polar moment of inertia, the mass density and the length of the shaft; φ and x are the rotation angle and the spatial co-ordinate along the shaft axis; and M_w and J are the equivalent resistance torque and the equivalent mass moment of inertia of the mechanism respectively. The equivalent moment of inertia J and the equivalent resistance torque M_w, which is assumed to be conservative, are periodic functions of φ. Dots and primes denote partial derivatives with respect to t and x respectively.

The motor torque M_m can be approximated around the operating point (M_0, Ω_0) as

$$M_m = M_0 + k(\Omega_0 - \Omega), \tag{4}$$

where Ω is the angular velocity of the motor, k is the slope of the motor characteristic around the operating point and M_0 and Ω_0 are the nominal motor torque and the nominal angular velocity respectively.

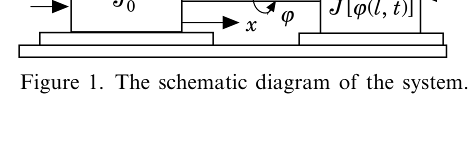

Figure 1. The schematic diagram of the system.

The boundary condition (3) can be linearized by introducing a new variable $\alpha_d(x, t)$:

$$\varphi(x, t) = \Omega_0 t + \left(-\frac{M_0}{GI_p} x + \frac{M_0}{GI_p} l\right) + \alpha_d(x, t). \tag{5}$$

If one inserts the transformation (5) into the wave equation (1) and the boundary conditions (2) and (3), neglects second and higher order terms and changes the time t to the non-dimensional variable $\tau = \Omega_0 t$, then one obtains

$$\alpha_{\tau\tau}(x, \tau) = (c^2/\Omega_0^2)\alpha_{xx}(x, \tau), \tag{6}$$

$$J_0\Omega_0^2\alpha_{\tau\tau}(0, \tau) + k\Omega_0\alpha_\tau(0, \tau) - GI_p\alpha_x(0, \tau) = 0, \tag{7}$$

$$J(\tau)\Omega_0^2\alpha_{\tau\tau}(l, \tau) + \frac{dJ(\tau)}{d\tau}\Omega_0^2\alpha_\tau(l, \tau) + \left[\frac{1}{2}\frac{d^2J(\tau)}{d\tau^2}\Omega_0^2 + \frac{dM_w(\tau)}{d\tau}\right]\alpha(l, \tau) + GI_p\alpha_x(l, \tau)$$

$$= -\frac{1}{2}\frac{dJ(\tau)}{d\tau}\Omega_0^2 - M_w(\tau). \tag{8}$$

Here, the index d in α_d is omitted, and subscripts x and τ denote associated partial derivatives [9, 10].

3. SERIES SOLUTION FOR THE FORCED RESPONSE

The forcing and coefficient functions of the inhomogeneous boundary condition (8) are periodic functions of τ. The forced response of a Hill's type differential equation under periodic forcing excitation is also periodic [11]. Then, the forced response of the wave equation (6) under the boundary conditions (7) and (8) is a periodic function of τ, and the classical Fourier analysis used for the solution of an ordinary differential equation with periodic coefficients can be extended to the present problem. Fortunately, the general solution of the wave equation derived by the separation of variables method can be used for this purpose.

The forced response of the wave equation (6) is assumed to be of the following form

$$\alpha(x, \tau) = \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\Omega_0}{c}x + B_n\cos\frac{n\Omega_0}{c}x\right)\sin n\tau + \left(C_n\sin\frac{n\Omega_0}{c}x + D_n\cos\frac{n\Omega_0}{c}x\right)\cos n\tau. \tag{9}$$

Notice that the series solution (9) satisfies equation (6) exactly and its value at $x = l$ forms a standard Fourier series for the boundary condition (8), including a differential equation with periodic coefficients.

Here, the unknown coefficients A_n, B_n, C_n and D_n denote the components of the amplitudes of the forced response. If the series solution (9) is inserted into the boundary condition (7), then an infinite system of linear homogeneous algebraic equations, and into the boundary condition (8) an infinite system of linear inhomogeneous equations, for the unknown coefficients are obtained. These two sets of equation are coupled and have to be solved simultaneously. Such a system can be solved in an approximate manner by taking the unknowns after the first N equal to zero and determining the first N unknowns from the first N equations. This approach is comparable to the series solution method used for numerical solution of partial differential equations [12]. It is also a combination method, of the Fourier analysis of an ordinary differential equation with periodic coefficients under periodic forcing excitation and the analysis of a continuous system excited periodically from a boundary [13, 14].

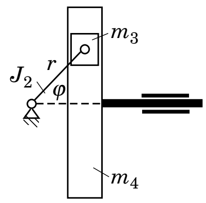

Figure 2. The Scotch-yoke mechanism.

4. EXAMPLE

The series solution method is now applied for a harmonic motion mechanism (see Figure 2). The equivalent moment of inertia of the mechanism is

$$J(\varphi) = a_0 + a_2 \cos 2\varphi, \tag{10}$$

where

$$a_0 = J_2 + m_3 r^2 + m_4 r^2/2, \qquad a_2 = -m_4 r^2/2. \tag{11}$$

The governing equation and the boundary conditions of the torsional vibrations of the drive shaft can be written as follows:

$$\alpha_{\tau\tau}(x, \tau) = (c^2/\Omega_0^2)\alpha_{xx}(x, \tau), \tag{12}$$

$$J_0 \Omega_0^2 \alpha_{\tau\tau}(0, \tau) + k\Omega_0 \alpha_\tau(0, \tau) - GI_p \alpha_x(0, \tau) = 0, \tag{13}$$

$$(a_0 + a_2 \cos 2\tau)\Omega_0^2 \alpha_{\tau\tau}(l, \tau) + (-2a_2 \sin 2\tau)\Omega_0^2 \alpha_\tau(l, \tau) + (-2a_2 \cos 2\tau)\Omega_0^2 \alpha(l, \tau) + GI_p \alpha_x(l, \tau) = a_2 \Omega_0^2 \sin 2\tau. \tag{14}$$

In the numerical calculations following parameter values are assumed:

$$J_0 = 0{\cdot}0155\ \mathrm{kg\,m^2}, \qquad D = 0{\cdot}03\ \mathrm{m}, \qquad a_0 = 0{\cdot}024\ \mathrm{kg\,m^2}, \qquad a_2 = -0{\cdot}004\ \mathrm{kg\,m^2},$$

$$l = 0{\cdot}3\ \mathrm{m}.$$

Maximum values of torsional vibrations at the mechanism side of the shaft for different motor characteristics are presented in Figure 3. It is shown that resonance appears for several value of the operation speed Ω_0. For the case of a soft motor ($k = 2$ Nms), resonance arises when the operation speed Ω_0 is equal to 754 rad/s. This is also equal to the half of the first frequency (1508 rad/s) of the free vibrations of a shaft having two disks

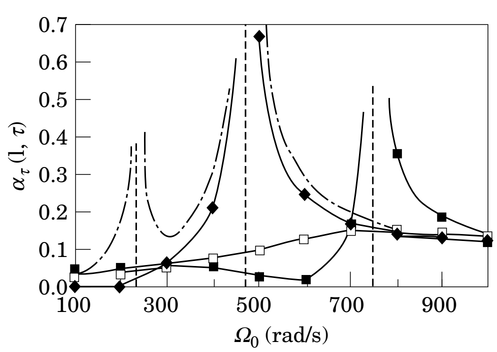

Figure 3. The vibration of the shaft on the mechanism side. –■–, $k = 2$; –□–, $k = 20$; –◆–, $k = 200$; —·—, $k = 2000$.

on two ends with the inertias J_0 and a_0. The motor side of the shaft can be assumed to be a free end in terms of relative rotations for the soft motor case. No resonance has been determined from the numerical solutions when the slope of the motor characteristic is equal to 20 Nms. There is a resonance point for $k = 200$ Nms when the operation speed is 472 rad/s. This is also equal to half of the first frequency (944 rad/s) of the free vibrations of a shaft which on the motor side is clamped and on the other has a disc with inertia a_0. The motor side of the shaft can be assumed to be a clamped end in that case. Two resonances arise for a very hard motor characteristic ($k = 2000$ Nms) when the operation speed is equal to both one half and one fourth of the first frequency of the free vibrations of the same system. These results are reminiscent of the properties of parametrically excited systems. Since the equivalent inertia function of the mechanism has only a second harmonic term, it may be concluded that these types of resonances are parametric ones mentioned by researchers in the literature. Similar results are also obtained for the second frequency of free vibrations, but have not been illustrated in the figure. Then, for the cases of soft and hard motor or a very large flywheel on the shaft, parametric resonance may occur and unstable regions may appear for a shaft modelled as a continuous element. The stability analysis seems hard work for continuous modelling of the shaft. However, 2π and 4π periodic forms of the series solution (9) may be a useful tool to determine the stability regions of the system by evaluating Hill's determinants for each boundary condition. However, it is not evident that this approach may be applied directly for the stability region determination. The general solution of the homogeneous boundary value problem has to be found by a detailed mathematical analysis. Such an analysis can probably be performed by completing the general solution of the wave equation given by D'Alembert and the solution of the ordinary differential equations with periodic coefficients in Floquet form.

5. PERTURBATION SOLUTION

The equivalent moment of inertia of the mechanisms and the resistance torques are generally in the form of small fluctuations around average values. In that case, a perturbation method can be used for the analysis of the forced response. In the previous paper, it was pointed out that this method requires some modifications in order to eliminate secular terms. The nominal angular velocity Ω_0 used in transformation (5) has to be changed to an unknown parameter Ω around the nominal value Ω_0.

One can define a small parameter ε in terms of the coefficients of the equivalent moment of inertia as $\varepsilon = a_1/a_0$. Most of the mechanisms have dominant second harmonic terms in their expansion of the equivalent inertia or resistance torque and some have no first harmonic terms. Then the small parameter ε has to be defined in terms of other coefficients.

Introducing a new variable $\alpha_d(x, t)$,

$$\varphi(x, t) = \Omega t + \left(-\frac{M_0}{GI_p}x + \frac{M_0}{GI_p}l\right) + \alpha_d(x, t), \tag{15}$$

expanding the angular velocity Ω around the nominal angular velocity Ω_0 as a perturbation series in ε,

$$\Omega = \Omega_0 + \varepsilon\Omega_1 + \varepsilon^2\Omega_2 + \cdots, \tag{16}$$

and changing to the non-dimensional time, τ, one can write the wave equation (1), and the boundary conditions (2) and (3) as follows:

$$\Omega^2\alpha_{\tau\tau}(x, \tau) = c^2\alpha_{xx}(x, \tau), \tag{17}$$

$$J_0\Omega^2\alpha_{\tau\tau}(0, \tau) + k\Omega\alpha_\tau(0, \tau) - GI_p\alpha_x(0, \tau) = -k(\varepsilon\Omega_1 + \varepsilon^2\Omega_2 + \cdots), \tag{18}$$

$$J(\tau)\Omega^2\alpha_{\tau\tau}(l, \tau) + \frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau}\Omega^2\alpha_\tau(l, \tau) + \left[\tfrac{1}{2}\frac{\mathrm{d}^2J(\tau)}{\mathrm{d}\tau^2}\Omega^2 + \frac{\mathrm{d}M_w(\tau)}{\mathrm{d}\tau}\right]\alpha(l, \tau) + GI_p\alpha_x(l, \tau)$$

$$= -\tfrac{1}{2}\frac{\mathrm{d}J(\tau)}{\mathrm{d}\tau}\Omega^2 - M_w(\tau). \tag{19}$$

One then seeks the solution of the wave equation (17) under the boundary conditions (18) and (19) as a perturbation series in ε:

$$\alpha(x, \tau) = \alpha_0(x,\tau) + \varepsilon\alpha_1(x, \tau) + \varepsilon^2\alpha_2(x, \tau) + \cdots. \tag{20}$$

If one inserts series (20) into equation (17) and the boundary conditions (18) and (19) and collects terms of like powers of ε, then one obtains the following boundary value problem ($i = 0, 1, 2$):

$$\begin{gathered}\Omega_0^2\,\partial^2\alpha_0(x, \tau)/\partial\tau^2 - c^2\,\partial\alpha_0(x, \tau)/\partial x = 0,\\ \Omega_0^2\,\partial^2\alpha_1(x, \tau)/\partial\tau^2 - c^2\,\partial\alpha_1(x, \tau)/\partial x = -2\Omega_0\Omega_1\,\partial^2\alpha_0(x, \tau)/\partial\tau^2,\\ \vdots\end{gathered} \tag{21.i}$$

$$\begin{gathered}\Omega_0^2J_0\frac{\partial^2\alpha_0(x, \tau)}{\partial\tau^2}\bigg|_{x=0} + k\Omega_0\frac{\partial\alpha_0(x, \tau)}{\partial\tau}\bigg|_{x=0} - GI_p\frac{\partial\alpha_0(x, \tau)}{\partial x}\bigg|_{x=0} = 0,\\ \Omega_0^2J_0\frac{\partial^2\alpha_1(x, \tau)}{\partial\tau^2}\bigg|_{x=0} + k\Omega_0\frac{\partial\alpha_1(x, \tau)}{\partial\tau}\bigg|_{x=0} - GI_p\frac{\partial\alpha_1(x, \tau)}{\partial x}\bigg|_{x=0} = -k\Omega_1\\ -2\Omega_0\Omega_1J_0\frac{\partial^2\alpha_0(x, \tau)}{\partial\tau^2}\bigg|_{x=0} - k\Omega_1\frac{\partial\alpha_0(x, \tau)}{\partial\tau}\bigg|_{x=0},\\ \vdots\end{gathered} \tag{22.i}$$

$$\begin{gathered}\Omega_0^2\frac{\partial^2\alpha_0(x, \tau)}{\partial\tau^2}\bigg|_{x=1} + \frac{GI_p}{a_0}\frac{\partial\alpha_0(x, \tau)}{\partial x}\bigg|_{x=1} = 0,\\ \Omega_0^2\frac{\partial^2\alpha_1(x, \tau)}{\partial\tau^2}\bigg|_{x=1} + \frac{GI_p}{a_0}\frac{\partial\alpha_1(x, \tau)}{\partial x}\bigg|_{x=1} = f_1\left[\alpha_0(x, \tau), \frac{\partial\alpha_0(x, \tau)}{\partial\tau^2}\bigg|_{x=1}\right.\\ \left.\frac{\partial^2\alpha_0(x, \tau)}{\partial\tau^2}\bigg|_{x=1}\Omega_0, \Omega_1, \tau\right]\\ \vdots\end{gathered} \tag{23.i}$$

Notice that the equation sets (21) and the boundary condition sets (22) and (23) include not only the coefficient functions α_i but also the components Ω_i. However, they are linear and enable one to solve for the components Ω_i and coefficient functions $\alpha_i(x, \tau)$ recursively. The homogeneous solutions should be eliminated at each step of the solution. In the first step only a constant term $-k\Omega_1$ appears on the right side of the boundary condition (22.1). Then the inhomogeneous solution of the boundary value problem (21.1), (22.1) and (23.1) is in the form $(-\Omega_1/\Omega_0)\tau$. Since the coefficient Ω_1 violates periodic solution forms, it should be eliminated. In the second step of the recursive solution again a constant term appears among the right side terms of the boundary condition (23.2). In that case, the particular solution of $\alpha_2(x, \tau)$ corresponding to this constant term will be in the form $s\tau + qx$. Then the second order component Ω_2 can be determined from the condition of the constant s being zero. At each step of the recursive solution, the particular solutions corresponding to the inhomogeneous harmonic terms can also be calculated by using an appropriate form of the solution assumption (9).

6. FINAL REMARKS

The forced response has been investigated by taking a more complex form of boundary condition on the motor side. Due to the effects of variation of the mechanism inertia, a resonance phenomenon appears without any external applied torque components for the cases of hard and soft motor characteristics. No resonance points have been determined from the numerical solutions for motor characteristics between these values. However, detailed studies are necessary to determine the stability of the system and the effects of the motor on possible unstable regions.

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